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LETTER TO THE EDITOR

Intermittency of growth and solitons of the nonlinear Schrödinger equation

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Abstract. Population explosions in systems with birth-death processes are studied. We show that the fluctuations in local birth and death rates are strongly enhanced by the exploding system and thus the intermittent spatial distribution of the reproducing particles is formed. It is characterized by presence of strong bursts separated by large areas with low population density. We estimate the probability of such rare bursts and find their typical forms. The analysis reveals a close mathematical relationship between the bursts and the solitons of the nonlinear Schrödinger equation.

In this letter we consider the statistical properties of spatio-temporal distributions which are formed in unstable systems undergoing an explosion. We limit our discussion to simple systems with reproduction, decay and diffusion, that are described by the following stochastic differential equation

$$\frac{\partial n}{\partial t} = \alpha n + f(x, t) n + D\frac{\partial^2 n}{\partial x^2}. \tag{1}$$

Here n is the population density of particles, α is the mean difference of their reproduction and death rates, and $f(x, t)$ is the fluctuating component of this difference.

The mathematical model (1) is found in many applications, such as chain nuclear and chemical reactions (Nicolis and Baras 1984) or population biology (May 1973, Goel and Richer-Dyn 1974, Zhang 1986, Mikhailov and Loskutov 1991). By a nonlinear transformation of the variable, equation (1) can be reduced to an equation that governs the ballistic growth of crystals (Kardar *et al* 1986). In the latter case, however, the function f is usually assumed to be time-independent (Nattermann and Renz 1989).

The explosion threshold is defined by the condition that above it the average density $\langle n \rangle$ begins to increase indefinitely in time. In absence of fluctuations, the threshold is reached when the reproduction rate becomes equal to the death rate, i.e. at $\alpha = 0$. Fluctuations decrease the threshold of explosion (Mikhailov and Uporov 1984, Mikhailov 1989).

It was noted (Zeldovich *et al* 1987) that the spatial distribution of a population which undergoes an explosion in a fluctuating medium should be highly non-uniform. It is characterized by presence of rare strong bursts which are separated by large regions with much lower density of the population. It was therefore proposed to call such distributions *intermittent.*

Hence, evolution of the spatial distribution of exploding populations may be approximately described in terms of rare strong bursts which wander through the medium, while continuing their exponential growth. As we shall see, in many aspects such bursts behave similar to an ensemble of independent particles.

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Since an individual burst is a rare statistical event, its most probable (i.e. *optimal*) form can be found by going to the path-integral solution for the probability functional of the random field $n(x, t)$ and by taking its appropriate optimal trajectory. As will be shown below, the variational equations for the optimal trajectories turn out to be very similar to the nonlinear Schrödinger equation (NSE). Moreover, individual bursts of the exploding population are closely related to solitons of NSE. To establish and investigate this relationship is the main aim of the present letter.

Below it is assumed that $f(x, t)$ in (1) represents a Gaussian noise and that

$$\langle f(x,t) f(x',t')\rangle = 2s(x-x')\delta(t-t') \tag{2}$$

where $s(x)$ specifies spatial correlations of the fluctuating death and reproduction rates. This function falls exponentially down to zero for separations x larger than the correlation radius r_0. The Stratonovich interpretation of the stochastic differential equation (1) is chosen.

The presence of intermittency becomes clear if we consider the equations for the multiple correlation functions

$$M_k(t, x_1, \ldots, x_k) = \langle n(x_1, t) \ldots n(x_k, t)\rangle. \tag{3}$$

Since (1) is linear, closed evolution equations for these quantities can be derived

$$\dot{M}_k = \alpha k M_k + \sum_{i,j=1}^{k} s(x_i - x_j) M_k + D \sum_{i=1}^{k} \frac{\partial^2}{\partial x_i^2} M_k. \tag{4}$$

It is convenient to write (4) in a slightly different form

$$\dot{M}_k = [\alpha + s(0)] k M_k - \hat{L} M_k \tag{5}$$

where

$$\hat{L} = -2 \sum_{\substack{i,j=1 \\ i<j}}^{k} s(x_i - x_j) - D \sum_{i=1}^{k} \frac{\partial^2}{\partial x^2}. \tag{6}$$

The first statistical moment $M_1(x, t) = \langle n(x, t)\rangle$ obeys a simple equation

$$\dot{M}_1 = [\alpha + s(0)] M_1 - D \frac{\partial^2}{\partial x^2} M_1. \tag{7}$$

If the spatial distribution of the population density is statistically translation-invariant and M_1 does not depend on the coordinate x, the last term in (7) vanishes. The first term in (7) describes the fluctuational shift of the explosion threshold. In the presence of fluctuations, the threshold is reached at $\alpha = -s(0)$.

For $k > 1$, the linear operator $\hat{L}$ is identical to the Hamiltonian operator for a system of k quantum particles which interact by a binary attractive potential. The general solution of (5) can be written as

$$M_k = \sum_l C_l \exp[(\alpha + s(0)) k t - \lambda_l t] \phi_l(x_1, \ldots, x_k) \tag{8}$$

where λ_l and ϕ_l are the eigenvalues and the eigenfunctions of the operator $\hat{L}$. For the continuous spectrum, the sum in (8) should be replaced by an integral.

Note that the bound states of particles, which correspond to negative energies in the related quantum mechanical problem, give rise to the exponentially growing contributions into functions M_k. In the long time limit the behaviour of M_k will be dominated by the contribution from the deepest 'energy level'.

If the correlation radius r_0 is small enough (i.e. $D \gg s(0) r_0$), we can approximately replace $s(x_i - x_j)$ by the delta function in the linear operator (6), i.e. to put

$$s(x_i - x_j) \approx \sigma \delta(x_i - x_j) \tag{9}$$

where

$$\sigma = \int s(x)\, \mathrm{d}x. \tag{10}$$

The exact spectrum of the linear operator $\hat{L}$ with function $s(x)$ given by (9) was found by Berezin *et al* (1964). Its minimal eigenvalue (that which corresponds to the deepest energy level in the quantum problem) is $\lambda_k = -\frac{1}{12}(\sigma^2/D)(k^3 - k)$. Consequently, the time dependence of the correlation functions in the long time limit will be

$$M_k \propto \exp[(\alpha + s(0))kt + (\sigma^2/12D)k(k^2 - 1)t]. \tag{11}$$

Fast relative increase of higher correlation functions indicates a progressive development of an intermittent spatial pattern.

Path-integral solutions of stochastic differential equations were introduced in various contexts by Martin *et al* (1973), Graham (1978), Janssen (1976) and Phytian (1977). Their construction for the case of stochastic reaction–diffusion equations was performed by Förster and Mikhailov (1986). In the special case of (1) with $s(x)$ given by (9) the probability functional for the random field $n(x, t)$ can be written in the form

$$P[n(x, t)] = \int \mathscr{D}p(x, t) \exp\left\{ \int \mathrm{d}x\, \mathrm{d}t (H - p\dot{n}) \right\} \tag{12}$$

where

$$H = \sigma(pn)^2 + \alpha pn + Dp \frac{\partial^2 n}{\partial x^2}. \tag{13}$$

Here $p(x, t)$ is an auxiliary field. Below we take $\alpha = 0$.

Optimal trajectories obey variational equations

$$\dot{n} = \frac{\partial H}{\partial p} \qquad \dot{p} = -\frac{\partial H}{\partial n}. \tag{14}$$

Explicitly, these equations read as

$$\dot{n} = D \frac{\partial^2 n}{\partial x^2} + 2\sigma p n^2 \tag{15}$$

$$\dot{p} = -D \frac{\partial^2 p}{\partial x^2} - 2\sigma p^2 n. \tag{16}$$

To specify an individual optimal trajectory, we should fix its initial and final points. For a distributed system this means fixing initial and final spatial distributions

$$n(x, t = 0) = n_0(x) \qquad n(x, t = T) = n_T(x). \tag{17}$$

If the transition from $n_0(x)$ to $n_T(x)$ represents an exponentially rare statistical event, its probability ρ can be estimated (up to some unknown pre-exponential factor) by taking the value of the probability functional (12) for the respective optimal trajectory. This yields

$$\rho \propto \exp\left[-\frac{1}{\sigma}\int \mathrm{d}x\,\mathrm{d}t(p_{\mathrm{opt}}n_{\mathrm{opt}})^2\right] \tag{18}$$

where $p_{\mathrm{opt}}(x, t)$ and $n_{\mathrm{opt}}(x, t)$ are the solutions of the variational equations (15).

The variational equations (15)-(16) are closely related to the nonlinear Schrödinger *equation* (NSE)

$$\mathrm{i}\frac{\partial\psi}{\partial t} = -\frac{\partial^2\psi}{\partial r^2} - 2\psi^2\psi^*. \tag{19}$$

It is well known (after Zakharov and Shabat 1971) that this equation is completely integrable. It supports a family of solutions

$$\psi = \frac{2\eta\exp[-4\mathrm{i}(\xi^2-\eta^2)t + 2\mathrm{i}\xi r + \mathrm{i}\delta]}{\cosh[2\eta(r-r_0) - 8\eta\xi t]} \tag{20}$$

where ξ, η, δ and r_0 are free parameters. They describe propagation of solitons with velocity $V = 4\xi$, width $(2\eta)^{-1}$ and amplitude 2η.

To reveal this analogy, we write the complex conjugate of (19),

$$\mathrm{i}\frac{\partial\psi^*}{\partial t} = \frac{\partial^2\psi^*}{\partial r^2} + 2(\psi^*)^2\psi. \tag{21}$$

Next, we consider ψ and ψ^* as two independent variables, $\psi \to u$ and $\psi^* \to v$, and perform the analytical continuation to imaginary time $t = -\mathrm{i}\tau$. Then equations (19) and (21) become

$$\dot{u} = \frac{\partial^2 u}{\partial r^2} + 2vu^2 \tag{22}$$

$$\dot{v} = -\frac{\partial^2 v}{\partial r^2} - 2v^2 u. \tag{23}$$

Clearly, both the nonlinear Schrödinger equations (19) and (21) with imaginary time and the variational equations (15), (16) represent the special cases of (22), (23). The difference is that, in the case of the NSE, variables u and v are complex and conjugated. In the case of equations (15), (16), both these variables are real.

Since we know the special soliton solution (20) of NSE, we can construct some special solutions of (22), (23) by analytical continuation in respect to time and the free parameters which are present in (20). Particularly, by putting $t = -\mathrm{i}\tau$, $\xi = \mathrm{i}\zeta$, $\delta = \mathrm{i}g$ in (20), we find

$$u = \frac{2\eta\exp[+4(\zeta^2+\eta^2)\tau - 2\zeta r - g]}{\cosh[2\eta(r-r_0) - 8\eta\zeta\tau]}. \tag{24}$$

The respective solution for v is constructed by first performing the analytical continuation to imaginary time $t = -\mathrm{i}\tau$ in the expression for ψ^* and then putting $\xi = \mathrm{i}\zeta$, $\delta = \mathrm{i}g$. This yields

$$v = \frac{2\eta\exp[-4(\zeta^2+\eta^2)\tau - 2\zeta r + g]}{\cosh[2\eta(r-r_0) - 8\eta\zeta\tau]}. \tag{25}$$

Now, if we assume that the parameters ζ, η and g in (24) and (25) are real, these expressions give the solution to coupled equations (22) and (23) in the case of real variables u and v. This can be verified, as well, by direct substitution of (24), (25) into equations (22), (23).

The variational equations (15), (16) are reduced to (22), (23) by simple rescaling of time and coordinate variables:

$$\tau=\sigma t \qquad r=(\sigma/D)^{1/2}x \tag{26}$$

and subsequent replacement of n by u and p by v. The same operations, applied to (24) and (25), yield certain special solutions to (15) and (16). They are discussed here.

First we consider a family of solutions obtained by putting $\zeta=0$ in (24) and (25). In the original variables x and t they read as

$$n=\frac{C\exp(qt)}{\cosh[(q/D)^{1/2}(x-x_0)]} \tag{27}$$

$$p=\frac{(q/C\sigma)\exp(-qt)}{\cosh[(q/D)^{1/2}(x-x_0)]}. \tag{28}$$

Here q and C are arbitrary constants; they can also be expressed in terms of η and g as $q=4\sigma\eta^2$, $C=2\eta\,e^{-g}$.

Equation (27) describes an *optimal* (i.e. most probable) burst that grows exponentially in time at a rate q. The centre of the burst is located at $x=x_0$ and does not change as time goes on. We see that the characteristic width δx of the burst is not an independent parameter, it is related to the growth rate q as $\delta x=(D/q)^{1/2}$. The narrower the burst, the larger its rate of growth.

Remarkably, the respective spatial distribution of the auxiliary field p repeats the profile of the staying burst (cf (27) and (28)). However, instead of growing, this field decays exponentially in time. Note that the product pn is time-independent.

The soliton solutions with $\zeta\neq 0$ give rise to the *travelling bursts* in the explosion problem. Using (24) and (25), we obtain

$$n=\frac{C\exp[-(V/2D)(x-Vt-x_0)]}{\cosh[(q/D)^{1/2}(x-Vt-x_0)]}\exp\{[q-(V^2/4D)]t\} \tag{29}$$

$$p=\frac{(q/C\sigma)\exp[(V/2D)(x-Vt-x_0)]}{\cosh[(q/D)^{1/2}(x-Vt-x_0)]}\exp\{-[q-(V^2/4D)]t\}. \tag{30}$$

Here we have

$$q=4\sigma\eta^2 \qquad V=4(\sigma D)^{1/2}\zeta \qquad C=2\eta\exp[g-(V/2D)x_0]. \tag{31}$$

Equation (29) describes bursts which, while growing exponentially in time, move along the coordinate axis x at a velocity V. When $V=0$, they represent the staying bursts that were considered above. In contrast to the staying bursts, the optimal travelling bursts are not symmetric. They are more steep in the front than in the rear side. The difference in the slopes increases with V. For each q, a critical velocity

$$V_c=2(qD)^{1/2} \tag{32}$$

exists. When V approaches V_c, the slope of the rear side of the travelling burst decreases and, at $V=V_c$, it becomes equal to zero. For higher values of V, equation (29) describes a solitary propagating front, with the population density n going to infinity as $x\to-\infty$.

The propagation velocity influences the rate Q of growth of a burst

$$Q = q - (V^2/4D). \tag{33}$$

It is interesting to note that Q vanishes precisely at the critical velocity V_c.

The associated pattern of the auxiliary field p, given by (30), represents a pulse which travels in the same direction with the burst of the population density n. However, its asymmetrical distortion is opposite to that found for the travelling burst. Namely, the pulse of the field p is more steep at its rear side than in the front. Moreover, the amplitude of this pulse is exponentially *decreasing* in time for $V < V_c$, in contrast to the behaviour found for the travelling burst.

The probability $\pi(Q, V, T)$ that a travelling burst with the velocity V and the growth rate Q is found in the system within the time interval T, can be estimated using (18). Substituting (29) and (30) into (18), integrating over x and applying (33), we obtain

$$\pi(Q, V, T) \propto \exp[-(4/3\sigma)(Q + V^2/4D)^{3/2} D^{1/2} T]. \tag{34}$$

For a given growth rate Q, the probability of bursts decreases with their propagation velocities V.

It should be emphasized that the stochastic partial differential equation (1) is *linear* and it cannot support the self-propagating patterns, which are known, for instance, in excitable media (Mikhailov 1990). Travelling or staying bursts represent rare statistical events which result from a random local increase of the birth rate (or a decrease of the death rate) that persists for some time period. The nonlinear equations (15), (16) are found in the variational problem and describe the most probable 'trajectory' which leads from a given initial population distribution $n(x, t = 0)$ to a given final distribution $n(x, t = T)$. The appearance of nonlinear terms in the variational equations (15), (16) can be traced to presence of a *multiplicative* noise in the original stochastic differential equation (1).

In this letter we have found only some special solutions of equations (15) and (16), which describe the bursts that are either staying or travelling at a constant velocity. A general solution of these equations would describe the most probable evolution leading from an arbitrary initial distribution $n(x, 0)$ to an arbirary final distribution $n(x, T)$. We have observed that the variational equations (15) and (16) are very closely related to the nonlinear Schrödinger equation. It is well known that the latter is completely integrable using the inverse scattering method. The same technique may be applied to obtain a general solution of (15) and (16).

The analogy between the variational equations (15), (16) and the nonlinear Schrödinger equations is not accidental. If we look at the path integral solution (12) for the probability functional, we can recognize further similarity between this statistical problem and quantization of the field which obeys in the classical limit the nonlinear Schrödinger equation. It was shown (Faddeev 1980) that the latter quantized field describes an ensemble of particles which interact by a contact binary attractive potential.

Above we considered only the one-dimensional problem. It was pointed out (Mikhailov 1989) that, for media of higher dimensionality, the statistical properties of population explosions are different. If the medium dimensionality exceeds two, intermittency is suppressed by strong enough diffusion.

Suppression of intermittency by diffusion does not mean that the bursts disappear. Rather, they become much more rare and, hence, do not give a dominant contribution into the lower correlation functions. The most probable individual bursts are described by the variational equations which can be obtained by a generalization of (15) and

(16) to the space of the respective dimensionality d. These equations are again closely related to the nonlinear Schrödinger equation.

It is known that for $d = 3$ the nonlinear Schrödinger equation does not permit the localized soliton solutions. Instead it describes wave collapses which occur within a finite time. After an appropriate continuation to imaginary times and complex values of the parameters, the collapse solutions of the NSE can be transformed into the burst solution of the explosion problem. The detailed analysis of intermittency in systems with $d > 1$ will be a subject of a separate publication.

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